Necessary Optimality Conditions for Infinite Horizon Control Problems under State Constraints

V. Basco[1], P. Cannarsa[2], and H. Frankowska[3]

Abstract: In some models of mathematical economics one encounters the following infinite horizon optimal control problem

$$V(t_0,x_0)=\inf\int_{t_0}^{\infty}e^{-\lambda t}\ell(x(t),u(t))\,dt$$

over all trajectory-control pairs (x,u), subject to the state equation under state constraints

$$\begin{cases} x'(t) &= f(t,x(t),u(t)), \quad u(t)\in U \quad \text{for a.e. } t\ge t_0 \\ x(t_0) &= x_0 \\ x(t) &\in K \quad \forall\, t\ge t_0, \end{cases}$$

where controls $u(\cdot)$ are Lebesgue measurable functions taking values in a given set U and $\lambda>0$. The literature addressing this problem deals with traditional questions of existence of optimal solutions, regularity of the value function V, necessary and sufficient optimality conditions. In the absence of state constraints, sufficient conditions for the local Lipschitz continuity of V are not difficult to obtain. The situation changes drastically when state constraints are present. Indeed, in this case there is even no guarantee that for every initial state $(t_0,x_0)\in R_+\times K$ there exists a (viable) trajectory of the control system satisfying also the state constraints. Even less can be said about the Lipschitz continuity of V.

While proving the existence of optimal controls requires only classical tools, recovering necessary conditions in the presence of state constraints and infinite horizon appears quite challenging, despite the fact that they are well investigated for the (finite horizon) Bolza problems under state constraints.

We provide here sufficient structural conditions to ensure local Lipschitz regularity of V. Then the dynamic programming is used to get the (Pontryagin) maximum principle under state constraints and also to derive partial and full sensitivity relations for the nonautonomous infinite horizon optimal control problem subject to state constraints. In economy, sensitivity relations lead to a significant economic interpretation. Indeed the co-state (of the maximum principle) can be regarded as the *shadow price* or *marginal price*, i.e. the contribution to the optimal total utility of a unit increase of capital x. Finally, under our assumptions, sensitivity relations also yield the classical transversality condition.

[1,2] Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Via della Ricerca Scientifica, 1 - 00133 Roma (Italy), *basco@mat.uniroma2.it, cannarsa@mat.uniroma2.it*

[3] CNRS and Institut de Mathématiques de Jussieu - Paris Rive Gauche, Université "Pierre et Marie Curie", case 247, 4 place de Jussieu, 75252 Paris (France) *helene.frankowska@imj-prg.fr*